Wavelet-based cross-correlation analysis of the scaling in molecular clouds

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Overview

We develop a wavelet-based cross-correlation (WCC) method to study the correlation between structural changes in molecular clouds as a function of scale. The method compares a pair of maps observed in different tracers or at different velocity ranges.

Advantages of the WCC method:
- Allows to measure the correlation coefficient and structural offset between two maps as a function of scale.
- Allows to weight individual pixels by their observational significance.
- Is robust against the noise.

Application of the WCC to simulated fBm (fractional Brownian motion) maps reveals that:
- Cross-correlation coefficient can strongly depend on the scale.
- Correlation coefficient and offset can be recovered robustly regardless of noise.

Analysis of the G333 molecular line maps (^{13}CO and C^{18}O) shows:
- A large scale gradient in the structural distribution. This could indicate a density structure where every core shows a low density tail towards the South-West mainly seen in ^{13}CO.

The WCC can be used to trace the correlated structural changes between different maps of a molecular cloud at scales representing the structural and physical importance such as chemical and phase transitions.

Wavelet cross-correlation (WCC)

Wavelet transform is proven to be a powerful tool for the scaling analysis in galaxies, interstellar clouds (Stutzki et al. 1998; Frick et al. 2001; Ossenkopf et al. 2008, hereafter O08). Convolution of the wavelet filter $\psi(l,\boldsymbol{r})$ ($\boldsymbol{r}$ =(x,y)) with an image $f(\boldsymbol{r})$ filters the image on a scale l given by the wavelet:

$$F(\boldsymbol{r},l) = \iint f(\boldsymbol{r})\psi(l,\boldsymbol{r})d\boldsymbol{r}.$$

Optimal wavelet filter for molecular clouds is found to be the Mexican-hat filter (O08), which provides the correct power spectral slope and spectral features:

$$\psi(\boldsymbol{r}) = \psi_c(\boldsymbol{r}) + \psi_a(\boldsymbol{r}) = \frac{4}{\pi l^2}\exp\left(\frac{\boldsymbol{r}^2}{(l/2)^2}\right) + \frac{4}{\pi l^2(v^2-1)}\left[\exp\left(\frac{\boldsymbol{r}^2}{(vl/2)^2}\right) - \exp\left(\frac{\boldsymbol{r}^2}{(l/2)^2}\right)\right],$$

where $\psi_c(\boldsymbol{r})$ and $\psi_a(\boldsymbol{r})$ are the core and annulus of the filter, and v =1.5 is the diameter ratio between the annulus and the core.

The **Δ-variance** (O08) of the filtered map $F(\boldsymbol{r},l)$ and its weights $w_F(\boldsymbol{r},l)$, indicating the significance of each pixel of the image, shows characteristic scales and the power spectral slope in the individual maps.

To study the cross-correlation of the amount of structure between two maps, f and g, on different scales, we introduce the **wavelet-based cross-power spectrum**,

$$C_l(\boldsymbol{\tau}) = \frac{C(l,\boldsymbol{\tau})}{\sigma_F(l)\,\sigma_G(l)} \qquad (1)$$

where

$$C(l,\boldsymbol{\tau}) = \iint_{x,y} \sqrt{w_F(\boldsymbol{r},l)w_G(\boldsymbol{r}+\boldsymbol{\tau},l)}\,F_w^*(\boldsymbol{r},l)\,G_w(\boldsymbol{r}+\boldsymbol{\tau},l)\,d\boldsymbol{r}.$$

$$\sigma_F(l) = \left(\iint_{x,y} w_F(\boldsymbol{r},l)\,F_w^2(\boldsymbol{r},l)\,d\boldsymbol{r}\right)^{1/2},\ \sigma_G(l) = \left(\iint_{x,y} w_G(\boldsymbol{r},l)\,G_w^2(\boldsymbol{r},l)\,d\boldsymbol{r}\right)^{1/2}$$

and

$$F_w = F(\boldsymbol{r},l) - \overline{F}_w(l), \quad G_w = G(\boldsymbol{r},l) - \overline{G}_w(l),$$

where

$$\overline{F}_w(l) = \frac{\iint w_F(\boldsymbol{r},l)F(\boldsymbol{r},l)\,d\boldsymbol{r}}{\iint w_F(\boldsymbol{r},l)\,d\boldsymbol{r}},\ \overline{G}_w(l) = \frac{\iint w_G(\boldsymbol{r},l)G(\boldsymbol{r},l)\,d\boldsymbol{r}}{\iint w_G(\boldsymbol{r},l)\,d\boldsymbol{r}}$$

are weighted means of two maps respectively.

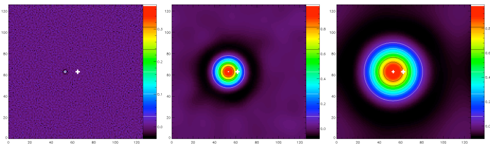
Cross-power spectrum (Eq. (1); see Fig 1) is used to estimate the **correlation coefficient** $r(l)$ on scale l,

$$r(l) = C_l(\boldsymbol{\tau} = 0) \qquad (2)$$

and **offset vector** $\boldsymbol{\tau}(l)$ = [Δx, Δy], where the Δx and Δy give the location of the correlation peak of cross-power spectrum

$$r_{\max}([\Delta x, \Delta y], l) = \arg\max_{x,y} C_l(\boldsymbol{\tau}).$$

Figure 1. Cross-power spectrum of two fBm maps (see, for example, Fig. 3, top panel) for scales of 1 pix, 15 pix, and 30 pix. Offset between fBm maps is 10 pix along the X-axis. The distance between the peak of the maximum correlation and the map center (white cross) shows the offset $\tau(l)$=10 pix between maps and the center of the cross-power spectrum gives the correlation coefficient $r(l)$ (see Eq. (2)).



Simple tests

We start testing the WCC for two simulated circular structures having Gaussian intensity profiles with amplitude=1 and standard deviations of 3 pix and 5 pix, which are offset by 5 pix along Y-axis (Fig. 2, top panel). The size of the Gaussians is traced by the maxima in the Δ-variance spectra (10-15 pix; 2nd panel). Correlation coefficient strongly depends on scale becoming large above the dominant structure scales. Amplitude and direction of the offset $\boldsymbol{\tau}(l)$ (bottom panel) are correctly recovered on those scales.

We also compared two fBm maps with spectral index of 3 (Fig. 3, top panel) where the second map is the filtered (maximum filter with a size of 15 pix) version of the first map, which mimics the opacity of optically thick lines. Correlation is negligible at small scales and it becomes significant at $l \geq 30$ pix (3rd panel). The correlation coefficient turns negative at scales below the mutual offset. The offset vector (length and direction) is exactly recovered for $l \geq 8$ pix (bottom panel).

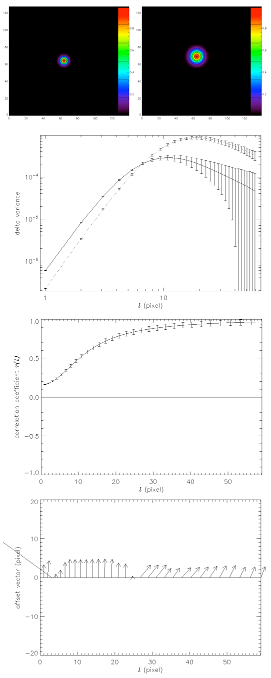


Figure 2. **Top panels.** Maps of Gaussian intensity profile **2nd panel**. Δ-variance of two maps (full and dotted lines represent the 1st and 2nd map, respectively)**. 3rd panel.** WCC function (full line): dependence of cross-correlation coefficient on scale.**. Bottom panel.** Offset vector as a function of scale: the vector shows the offset with respect to the first image.

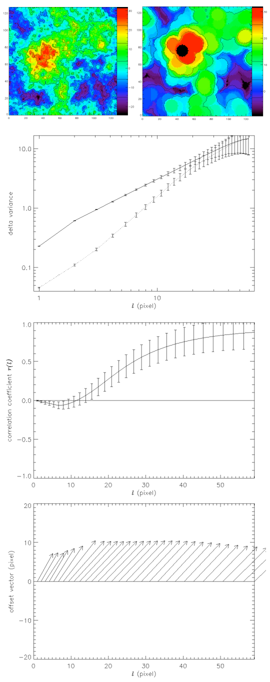
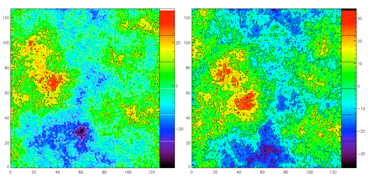


***Figure 3*.** Analysis for original fBm map (left) and the same map filtered on sclae of 15 pix and shifted for 14 pix in Nord-West direction (10 pix along X-axis and 10 pix along Y-axis). Description of panels as in Fig. 2.

Realistic tests

In reality the offset can vary on different spatial scales. To test this, we generate a fBm map with S/N=10 (Fig. 4, top left panel) and use the Fourier shift theorem to shift the large scales ($l \geq 10$ pix) by 18 pix in the South-West direction (top right panel). The Δ-variance spectrum has a bump at small scales due to low S/N value (2nd panel). Application of the WCC for these two maps shows that the offset vector can be recovered robustly (bottom panel). Structures are correlated significantly at small scales (l < 10 pix) as expected from the construction of the maps. The correlation gradually decreases from 10 to 20 pix (3rd panel) due to the shift of a structure at scales l > 10 pix. At scales of ≥ 20 pixels the structure sizes exceed the offset so that we find a gradual increase of the correlation coefficient at large scales.



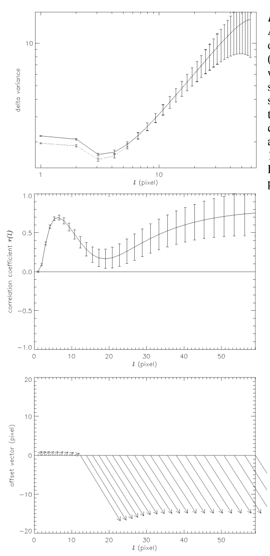
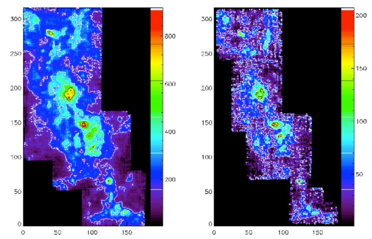


Figure 4. Application for original fBm map (left) and fBm map with structure on sclaes l >10 pix shifted by 18 pix in the South-West direction (15 pix along X-axis and 10 pix along Y-axis). Description of panels as in Fig. 2.

Application to GMC G333

We applied the WCC to the maps of the giant molecular cloud (GMC) G333 observed in ^{13}CO and C^{18}O 1-0 (Lo et al. 2009; Fig. 5, top panels). Their Δ-variance spectra have pronounced structure on scales of about 30 pix and 50 pix respectively and the structure is strongly correlated at scales > 5 pix (3rd panel). Large differences between the maps only occur at small scales affected by noise. The structure is offset at scales larger than 40 pix with amplitude of ~ 5 pix along the filament, where all structures in ^{13}CO are shifted to the South-West relative to C^{18}O. This could indicate a large-scale column density gradient enhancing the structure in ^{13}CO in that direction.



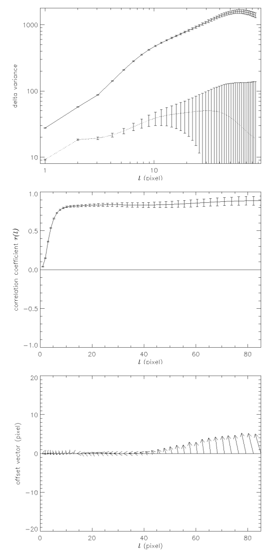


Figure 5. Application for molecular line maps of the GMC G333: ^{13}CO and C^{18}O (left and right respectively). The cloud is at a distance of 3.7 kpc and covers the region of sky of 0°.6 × 1°.2 (40 pc × 80 pc). The beam resolution = 0°.01 (2.7 pix) and spatial resolution is ~1 pc. Description of panels as in Fig. 2.